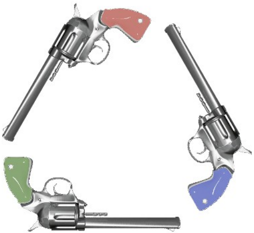
The Lanchester Truel

Attrition Dynamics of Multilateral War



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Naval Postgraduate School
Monterey, California

The University of York

ISMOR, RHUL, July 2017

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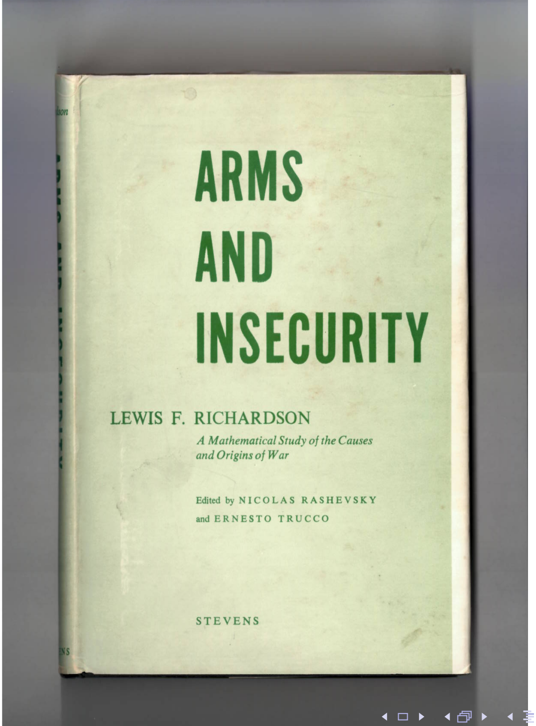
Lanchester's laws — Lanchester, 1913-1916


ARMS
AND
INSECURITY
LEWIS F. RICHARDSON
A Mathematical Study of the Causes and Origins of War
Edited by NICOLAS RASHEVSKY
and ERNESTO TRUCCO
STEVENS

Multilateral stability

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Richardson on 3 nations:
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On N nations:
'the world will for most of the time be content with just enough stability'

Triadic stability

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'the triadic situation often favors the weak over the strong'

Caplow, 1956, Coalitions in the triad

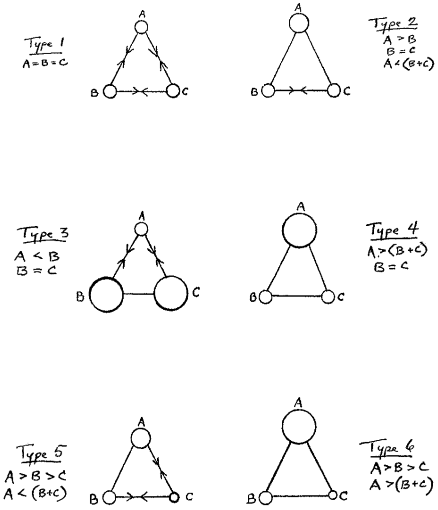
Type 1
A = B = C
A
B
C
A
B
C
Type 2
A > B
B = C
A < (B+C)
Type 3
A < B
B = C
A
B
C
A
B
C
Type 4
A > (B+C)
B = C
Type 5
A > B > C
A < (B+C)
A
B
C
A
B
C
Type 6
A > B > C
A > (B+C)

FIGURE 1

The Truel

The sequential truel:
A, B, C shoot, with hitting probabilities a, b, c such that $a > b > c$.

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Better marksmanship can hurt!

Brams and Kilgour, 1997, The Truel

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Some conclusions are robust:
the weakness of being the best marksman, the fragility of pacts.

Often these conclusions are counterintuitive or paradoxical.

The Lanchester Duel

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The line trajectory $\sqrt{a}A = \sqrt{b}B$ results in mutual annihilation.

The Lanchester Truel

$$\frac{dA}{dt} = -b(1-\beta)B - c\gamma C$$

$$\frac{dB}{dt} = -a\alpha A - c(1-\gamma)C$$

$$\frac{dC}{dt} = -a(1-\alpha)A - b\beta B$$

$a > b > c > 0, \quad 0 \leq \alpha, \beta, \gamma \leq 1.$

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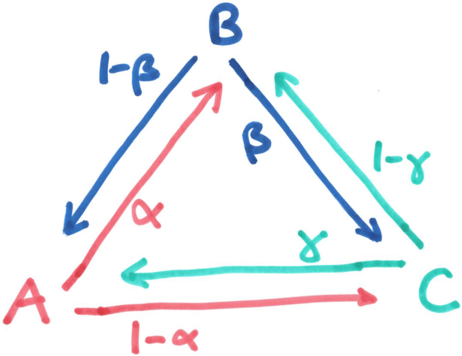
What happens next?


B
1-β
β
1-γ
α
γ
A
C
1-α

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The Lanchester Truel

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- kill rates which depend on opponent

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Here we keep it simple.

The Lanchester Truel

$$\begin{aligned}
\dot{A} &= && -b(1-\beta)B && -c\gamma C \\
\dot{B} &= -a\alpha A && && -c(1-\gamma)C \\
\dot{C} &= -a(1-\alpha)A && -b\beta B &&
\end{aligned}$$

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The decision parameters are α (for A), β for B, γ for C.
These need not be fixed but are typically dynamical, varying.

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We seek a Nash equilibrium or, failing that, an adaptive dynamics on (α, β, γ).

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If the **objective function** for each player is
its numbers minus others' numbers, *e.g.* (for A) $A_\infty - B_\infty - C_\infty$,

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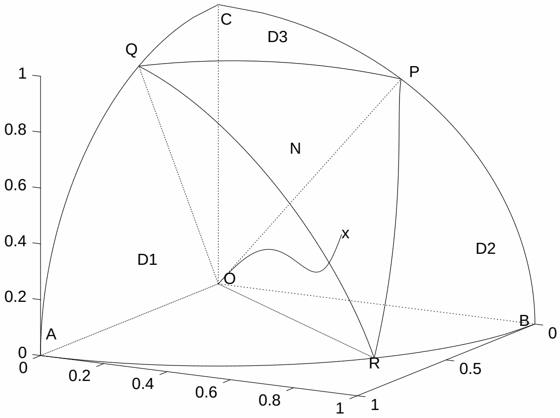
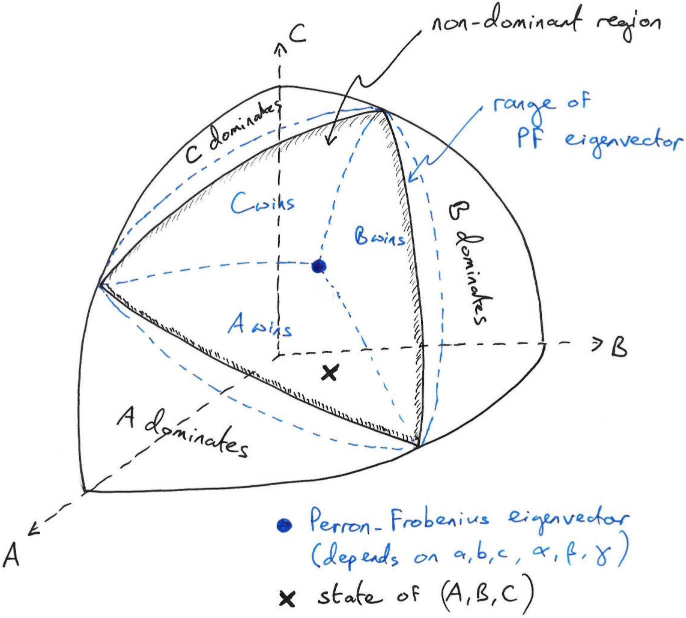
either one force can beat the other two together,

or the outcome is mutual annihilation

The Lanchester Truel




C
non-dominant region
range of PF eigenvector
C dominates
C wins
B wins
B dominates
A wins
B
A dominates
A
Perron-Frobenius eigenvector (depends on a, b, c, α, β, γ)
state of (A, B, C)

Lemma 1: The range of • encloses the non-dominant region, with equality when $a = b = c$.
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Then • simply chases the state $\times$*.*

The Lanchester Truel

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Suppose that the only thing a force values is reducing its own casualty rate:

A wants to maximize $\ddot{A}$, likewise for B and C.

Impose a rapid, adaptive dynamics on α, β, γ:

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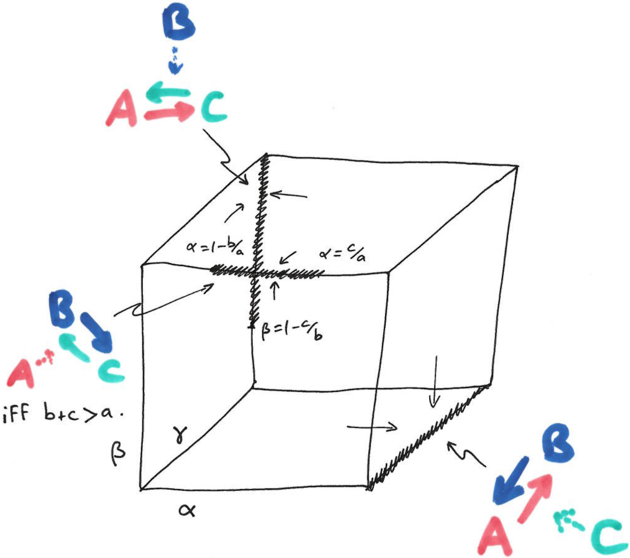
$$\frac{1}{\tau}\frac{d\beta}{dt} = c(1-\gamma) - a\alpha \tag{2}$$

$$\frac{1}{\tau}\frac{d\gamma}{dt} = a(1-\alpha) - b\beta. \tag{3}$$


B
A
C
α=1−b/a
α=c/a
β=1−c/b
B
A
C
iff b+c>a.
γ
β
α
B
A
C

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short-term **tactical**: maximize $\ddot{X}$, the rate of reduction of X's casualty rate,
then fire distributions approach stable states in which two players target only each other,
and the weakest player has an advantage because it is least capable of hurting the others.

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Thank you for listening.